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Fundamental groups having the whole information of spaces

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Abstract

We introduce a new construction of spaces from groups using homomorphic images of the fundamental group of the Hawaiian earring $\mathbb{H}$. According to this construction the Menger sponge, Sierpinski gasket, Sierpinski carpet and the direct product of their countable many copies are recovered from their fundamental groups.

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1. Introduction and definitions

Fundamental groups are defined for arbitrary spaces and are isomorphic among spaces of the same homotopy type. Therefore we usually think the fundamental group of a space has less information than that of the space itself. Is there a space whose fundamental group has the same information as that of the space itself? In the present paper we propose a new construction of a space from a group and show that this construction from the fundamental groups of certain spaces produces the original spaces, which implies that the fundamental groups of such spaces have the same information as that of the topological spaces themselves. The roots of this investigation are in the following [2, Theorem 1.3]:

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> Let X and Y be one-dimensional, locally path-connected, path-connected, metric spaces which are not semi-locally simply connected at any point. If the fundamental groups of X and Y are isomorphic, then X and Y are homeomorphic.

As our construction of spaces is based on this fact, the topologies of one dimensional fractals which are Peano continua and have non-trivial fundamental groups, e.g., the Sierpinski gasket, the Sierpinski carpet and the Menger sponge, are recovered from their fundamental groups according to this construction. We shall see that finite or countable products of such spaces are also recovered from their fundamental groups.

In the remaining part of this section we state basic definitions. The Hawaiian earring is the plane continuum $\mathbb{H} = \bigcup_{n=1}^{\infty}\{(x, y)\colon\ (x + 1/n)^2 + y^2 = 1/n^2\}$ and o denotes the origin $(0, 0)$. Each simple closed curve of the Hawaiian earring is parametrized as follows: $\mathbf{e}_n(t) = ((\cos(2\pi t) - 1)/n, \sin(2\pi t)/n)$ for $1 \leqslant n < \omega, 0 \leqslant t \leqslant 1$. (Here, $\mathbf{e}_n$ refers to the nth earring, that is the nth simple closed curve.) The fundamental group $\pi_1(\mathbb{H})$ is isomorphic to the free σ-product $\times_{n<\omega}\mathbb{Z}_n$, where each $\mathbb{Z}_n$ is a copy of the integer group $\mathbb{Z}$. We refer the reader to [3] for the notion of infinitary words and a free σ-product $\times^{\sigma}_{i\in I} G_i$. A locally path-connected, connected, compact metric space is called as a Peano continuum.

2. Construction of spaces and n-slender groups

The fundamental group of the Hawaiian earring $\pi_1(\mathbb{H})$ plays a principal role in this construction. The group $\pi_1(\mathbb{H})$ is presented algebraically as $\times_{n<\omega}\mathbb{Z}_n$ (abbreviated by $\times_\omega\mathbb{Z}$) [6,9,3,1]. Let δ_n denote a generator of $\mathbb{Z}_n$ which corresponds to a winding $\mathbf{e}_n$ to the nth circle of the Hawaiian earring.

Let $\mathcal{H}_G$ be the set of all subgroups of a group G which are homomorphic images of $\times_\omega\mathbb{Z}$. We remark that finitely generated subgroups are homomorphic images of $\times_\omega\mathbb{Z}$, but infinitely generated free groups are not homomorphic images of $\times_\omega\mathbb{Z}$. Moreover, any free product $*_{i\in I} G_i$ with infinitely many non-trivial factors is not a homomorphic image of $\times_\omega\mathbb{Z}$ by [3, Corollary 2.2]. (See Theorem 4.1 for a related result.)

The notation "$F \Subset X$" means "F is a finite subset of X". A finite subset F of $\mathcal{H}_G$ said to be *compatible*, if there exists $H \in \mathcal{H}_G$ such that $\bigcup F \subseteq H$. For $H \in \mathcal{H}_G$ and $F \Subset \mathcal{H}_G$ H is compatible with F, if $\{H\} \cup F$ is compatible. We say a subfamily C of $\mathcal{H}_G$ is compatible, if any finite subset of C is compatible. Let X_G be the set of all maximal compatible non-empty subfamilies of $\mathcal{H}_G$ which contain an uncountable subgroup.

Since $\times_\omega\mathbb{Z} \simeq F * \times_\omega\mathbb{Z}$ for a finitely generated free groups F, $x \in X_G$ is closed under conjugacy, that is, $u^{-1}Hu \in x$ for $H \in x$ and $u \in G$. In addition, $x \in X_G$ is downward-closed, that is, $H \in x$ and $H' \leqslant H$ imply $H' \in x$ for $H' \in \mathcal{H}_G$.

Since every compatible family extends to a maximal compatible family, X_G is non-empty if and only if $\mathcal{H}_G$ contains an uncountable subgroup. X_G is a one-point set or empty, if G is a homomorphic image of $\times_\omega\mathbb{Z}$.

For subgroups H, H' of G, $H \preceq H'$ if there exists $F \Subset G$ such that $H \leqslant \langle H' \cup F\rangle$ holds. If H is finitely generated, then $H \preceq \{e\}$ holds.

The first proposition shows the root of the above definition.

Proposition 2.1. *Let X be a path-connected, one-dimensional metric space and G be the fundamental group $\pi_1(X, x_0)$. Then each uncountable $H \in \mathcal{H}_G$ belongs to a unique $x \in X_G$.*

Proof. Let $h : \times_\omega \mathbb{Z} \to H$ be a surjective homomorphism. By [2, Theorem 1.1] we have a unique point $x^* \in X$, a path p from x^* to x_0 and a continuous map $f : (\mathbb{H}, o) \to (X, x^*)$ such that $h = \varphi_p \cdot f_*$ where φ_p is a base-point-change isomorphism from $\pi(X, x^*)$ to $\pi(X, x_0)$ according to p.

First we show that for uncountable $H, K \in \mathcal{H}_G$, $\{H, K\}$ is compatible if and only if the point x^* determined by H and y^* determined by K are the same. To see this, suppose that $x^* = y^*$. Then we have continuous maps $f : (\mathbb{H}, o) \to (X, x^*)$ and $g : (\mathbb{H}, o) \to (X, x^*)$ and paths p and q from x^* to x_0 so that $H = \mathrm{Im}(\varphi_p \cdot f_*)$ and $K = \mathrm{Im}(\varphi_q \cdot g_*)$. Define $f' : (\mathbb{H}, o) \to (X, x^*)$ so that $f' \cdot \mathbf{e}_{2n} = f \cdot \mathbf{e}_n$ and $f' \cdot \mathbf{e}_{2n+1} = g \cdot \mathbf{e}_n$. Then $\langle [p^- q] \rangle * \mathrm{Im}(\varphi_p \cdot f'_*) \in \mathcal{H}_G$ contains both H and K, which shows that H and K are compatible. Conversely suppose that H and K are compatible. Then we have $H' \in \mathcal{H}_G$ such that $H, K \leqslant H'$. Let $g' : (\mathbb{H}, o) \to (X, z^*)$ be a continuous map and p' be a path from z^* to x_0 such that $\mathrm{Im}(\varphi_{p'} \cdot g'_*) = H'$. Suppose that x^* is not equal to z^*. Then g' maps almost all the circles of $\mathbb{H}$ in a neighborhood of z^*. On the other hand there are uncountably many essential loops in a neighborhood of x^*, which is a contradiction. We refer the reader to the proof of [2, Lemma 4.2] for a precise proof to deduce the contradiction. This fact implies that an uncountable $H \in \mathcal{H}_G$ belongs to a unique $x \in X_G$. □

We introduce a topology on the set X_G as follows.

For $Y \subseteq X_G$, Y is closed if each point x with the following property belongs to Y; there exists a sequence $(x_n\colon n < \omega)$ of elements of Y satisfying the following condition $(*)$.

$(*)$ For given uncountable $H_n \in x_n$ $(n < \omega)$ there exist $H'_n \in x_n$ satisfying the following:

- $H_n \preceq H'_n$;
- for arbitrary $a_n \in H'_n$ $(n < \omega)$ there exists $h : \times_{n<\omega} \mathbb{Z}_n \to G$ such that $h(\delta_n) = a_n$ for every $n < \omega$ and $\mathrm{Im}(h) \in x$.

In general X_G may not satisfy any separation axiom, but we write $\lim_{n\to\infty} x_n = x$ when $(*)$ holds. If G is the fundamental group $\pi_1(X, x_0)$ for a one-dimensional Peano continuum X, then $(*)$ is equivalent to each of the following simpler statements, i.e.,

(T1) for given uncountable $H_n \in x_n$ $(n < \omega)$, there exist $e \neq a_n \in H_n$ and $g_n \in G$ such $\{g_n^{-1} a_n g_n\colon n < \omega\} \subseteq H$ for some $H \in x$;

(T2) there exist uncountable $H_n \in x_n$ $(n < \omega)$ such that for arbitrary $a_n \in H_n$ $(n < \omega)$ there exists $H \in x$ such that $\{a_n\colon n < \omega\} \subseteq H$.

Each of these is equivalent to the condition $(*)$, because an uncountable $H \in \mathcal{H}_G$ determines a point of X for a one-dimensional, locally path-connected, path-connected, metric space. Therefore we need not use the notion of maximal compatible families as well. However, it seems to be necessary to use maximal compatible families and $(*)$ to introduce a reasonable topology.

Theorem 2.2. *Let A be an Abelian group. Then X_A is an empty or one-point space.*

Proof. Let $B \leqslant A$ be a homomorphic image of $\times_\omega \mathbb{Z}$. Then B is a homomorphic image of the abelianization of $\times_\omega \mathbb{Z}$, where the abelianization of $\times_\omega \mathbb{Z}$ is isomorphic to $\mathbb{Z}^\omega \oplus C$ for some algebraically compact group C [5]. Here, $\mathbb{Z}^\omega \simeq \mathbb{Z}^\omega \oplus \mathbb{Z}^\omega$ and $C \simeq C \oplus C$ for this C. Therefore $\mathcal{H}_A$ itself is a compatible family and hence X_A is an empty or one-point space. $\square$

As our construction of spaces X_G shows, our concern is concentrated to uncountable subgroups of a given group G. For our purpose we are interested in a certain class of groups. We recall n-slender groups from [3, Section 3] together with [3, Proposition 3.2].

Definition 2.3. A group S is *n-slender* if, for each homomorphism $h : \times_{n<\omega} \mathbb{Z}_n \to S$, the set $\{n < \omega\colon\ h(\delta_n) \neq e\}$ is finite.

The class $\mathcal{S}$ consists of all the groups G such that, for any non-trivial element $g \in G$, there exist an n-slender group S and a homomorphism $h : G \to S$ with $h(g) \neq e$.

Since a finite direct product of n-slender groups is again n-slender [3, Theorem 3.6], we have,

Proposition 2.4. *The following conditions are equivalent for a group G:*

(1) *G belongs to $\mathcal{S}$*;
(2) *G is a subgroup of a direct product of n-slender groups*;
(3) *G is a subgroup of an inverse limit of n-slender groups.*

Lemma 2.5. *Let $G \in \mathcal{S}$ and h and h' be homomorphisms from $\times_{n<\omega} \mathbb{Z}_n$ to G. If $h(\delta_n) = h'(\delta_n)$ for every $n < \omega$, then $h = h'$.*

Proof. Let S be an n-slender group and $g : G \to S$ a homomorphism. There exists $m < \omega$ such that $g \cdot h(\times_{n \geqslant m} \mathbb{Z}_n) = g \cdot h'(\times_{n \geqslant m} \mathbb{Z}_n) = \{e\}$. Hence $g \cdot h = g \cdot h'$ by the assumption. $\square$

Lemma 2.6. *Let $G \in \mathcal{S}$ be a homomorphic image of $\times_{n<\omega} \mathbb{Z}_n$. Then G is finitely generated or uncountable.*

Proof. Let $h : \times_{n<\omega} \mathbb{Z}_n \to G$ be a surjective homomorphism. Suppose that $h(\delta_n) = e$ for $n > m$. Since there exists $h_0 : \times_{n<\omega} \mathbb{Z}_n \to G$ such that $h_0(\delta_n) = h(\delta_n)$ for every $n < \omega$ and $\mathrm{Im}(h_0) = \langle h_0(\delta_0), \ldots, h_0(\delta_m) \rangle$, G is finitely generated by Lemma 2.5. Otherwise, we choose a strictly increasing sequence i_n $(n < \omega)$ and homomorphisms g_n from G to n-slender groups S_n by induction so that $g_n h(\delta_{i_n}) \neq e$ and $g_k h(\delta_{i_n}) = e$ for $k < n$. For $f : \omega \to \{0, 1\}$, define $W_f = \delta_{i_0}^{f(0)} \delta_{i_1}^{f(1)} \cdots \delta_{i_n}^{f(n)} \cdots$, where $\delta_{i_k}^0$ is an empty word. Then $h(W_f)$ and $h(W_{f'})$ are distinct for $f \neq f'$, which implies that G has the continuum cardinality. $\square$

Proposition 2.7. *Let S_i $(i \in I)$ be n-slender groups. Then X_G is an empty or one point space in the following cases*:

(1) *G is the inverse limit of n-slender groups. In particular, $G = \varprojlim(*_{i\in F} S_i\colon F \Subset I)$*;
(2) *$G = \bigtimes^{\sigma}_{i\in I} S_i$.*

Proof. We first prove the conclusion for the case (1). Let $G = \varprojlim(T_j, p_{ij}\colon i, j \in J)$ and $p_j : G \to T_j$ be the projection. Let $h_0, h_1 : \bigtimes_{n<\omega}\mathbb{Z}_n \to G$ be homomorphisms. Define $h : *_{n<\omega}\mathbb{Z}_n \to G$ by $h(\delta_{2n}) = h_0(\delta_n)$ and $h(\delta_{2n+1}) = h_1(\delta_n)$. Since T_j is n-slender, $p_j \cdot h$ extends on $\bigtimes_{n<\omega}\mathbb{Z}_n$ uniquely and consequently h extends on $\bigtimes_{n<\omega}\mathbb{Z}_n$ uniquely by the universal property of the inverse limit. Now we have $\mathrm{Im}(h_0) \cup \mathrm{Im}(h_1) \subseteq \mathrm{Im}(h)$, which implies that each two elements of $\mathcal{H}_G$ are compatible and hence X_G is an empty or one-point space. Since the class of n-slender groups is closed under taking free products [3, Theorem 3.6], the second statement of (1) is a special case of the first one.

To show the conclusion for the second case, we recall standard homomorphisms from [4]. For $G = \bigtimes^{\sigma}_{i\in I} S_i$, let $h_0, h_1 : \bigtimes_{n<\omega}\mathbb{Z}_n \to G$ be homomorphisms. By [4, Theorem 2.3], there exist $u_0, u_1 \in G$ and standard homomorphisms $\bar{h}_0, \bar{h}_1 : \bigtimes_{n<\omega}\mathbb{Z}_n \to G$ such that $h_0 = u_0^{-1}\bar{h}_0 u_0$ and $h_1 = u_1^{-1}\bar{h}_1 u_1$ hold.

Then, according to [3, Proposition 1.9] and [4, Lemma 2.4], $h : \bigtimes_{n<\omega}\mathbb{Z}_n \to G$ is uniquely determined by the equations $h(\delta_{2n}) = \bar{h}_0(\delta_n)$ and $h(\delta_{2n+1}) = \bar{h}_1(\delta_n)$. Now we have $u_0 \,\mathrm{Im}(h_0) u_0^{-1} \cup u_1 \,\mathrm{Im}(h_1) u_1^{-1} \subseteq \mathrm{Im}(h)$, which implies that X_G is an empty or one-point space. □

Remark 2.8.

(1) In the definition of X_G we use maximal compatible families and do not use directed families. For $H, H' \in x \in X_G$ there may not exist $K \in x$ such that $H \cup H' \subseteq K$, there exists $K' \in \mathcal{H}_G$ such that $H \cup H' \subseteq K'$. If we restrict $x \in X_G$ to be directed, it is unclear whether a directed family extends a maximal one. Therefore we adopt maximal compatible families.
(2) Our construction of topological spaces from groups is not categorical. Homomorphisms may not induce continuous maps. Moreover, even when a continuous map is induced, the injectivity of a homomorphism does not imply that of the induced continuous map [2, Remark 6.9 (2)].

3. Direct products

In this section we are concerned with direct products of groups. Since the fundamental group of a direct product of path-connected spaces is the direct product of those of components, one may expect that our construction may commute with direct products. We show this is indeed the case under certain conditions.

Lemma 3.1. *Let $H = \prod_{m<\omega} H_m$ where each H_m is a homomorphic image of $\times_{n<\omega}\mathbb{Z}_n$. Then H is also a homomorphic image of $\times_{n<\omega}\mathbb{Z}_n$.*

Proof. It suffices to show that $\prod_{m<\omega} \times_{n<\omega}\mathbb{Z}_{(m,n)}$ is an homomorphic image of $\times_{(m,n)\in\omega\times\omega}\mathbb{Z}_{(m,n)}$, where $\mathbb{Z}_{(m,n)}$'s are copies of $\mathbb{Z}$. To define a homomorphism we need a notion of infinitary words and other ones from [3, p. 244]. $\mathcal{W}(\mathbb{Z}_{(m,n)}\colon m,n<\omega)$ is the set of words, that is, infinitary words, whose letters are in $\mathbb{Z}_{(m,n)}$'s. For a word W and a subset I of $\omega\times\omega$, W_I is a word obtained from W by deleting letters not belonging to $\bigcup_{(m,n)\in I}\mathbb{Z}_{m,n}$. Define $h : \times_{(m,n)\in\omega\times\omega}\mathbb{Z}_{(m,n)} \to \prod_{m<\omega}\times_{n<\omega}\mathbb{Z}_{(m,n)}$ by: $h(W)(m) = W_{I_m}$, where $I_m = \{(m,n)\colon n<\omega\}$. Then h is a surjective homomorphism. □

In the sequel of this section, when $G = \prod_{m<\omega} G_m$, let p_m denote the projection from G to G_m for each m.

Lemma 3.2. *Let $G = \prod_{m<\omega} G_m$ and $x \in X_G$. Then $\prod_{m<\omega} p_m(H') \in x$ for $H' \in x$ and consequently $x = \{H \in \mathcal{H}_G\colon H \leqslant \prod_{m<\omega} p_m(H')$ for $H' \in x\}$ holds.*

Proof. Since $p_m(H) \in \mathcal{H}_{G_m}$ for $H \in x$ and each $\{p_m(H)\colon H \in x\}$ is compatible, $\{H \in \mathcal{H}_G\colon H \leqslant \prod_{m<\omega} p_m(H')$ for $H' \in x\}$ is compatible and contains x by Lemma 3.1. We have the conclusion by the maximality. □

Lemma 3.3. *Let $H_0 \preceq K_0$ and $H_1 \preceq K_1$. Then $H_0 \times H_1 \preceq K_0 \times K_1$ holds.*

Proof. There exist $F_0 \Subset G_0$ and $F_1 \Subset G_1$ such that $H_0 \leqslant \langle K_0 \cup F_0\rangle$ and $H_1 \leqslant \langle K_1 \cup F_1\rangle$. Now $H_0 \times H_1 \leqslant \langle K_0 \times K_1 \cup F_0 \times \{e\} \cup \{e\} \times F_1\rangle$ and we have the conclusion. □

Lemma 3.4. *Let $G = G_0 \times G_1$ and $x \in X_G$. If $H \times \{e\} \in x$ and $\{e\} \times K \in x$, then $H \times K \in x$.*

Proof. Let $F \Subset x$. Since the family $\{X_0 \times X_1\colon X_0 \in \mathcal{H}_{G_0},\ X_1 \in \mathcal{H}_{G_1}\}$ is cofinal in $\mathcal{H}_G$, we can find $H_0 \times H_1$ and $K_0 \times K_1$ with $H_i, K_i \in \mathcal{H}_{G_i}$ so that $H \times \{e\} \cup \bigcup F \subseteq H_0 \times H_1$ and $\{e\} \times K \cup \bigcup F \subseteq K_0 \times K_1$. Since $\bigcup F \subseteq (H_0 \cap K_0) \times (H_1 \cap K_1)$, $(H \times K) \cup \bigcup F \subseteq H_0 \times K_1 \in \mathcal{H}_G$. We have $H \times K \in x$ by the maximality of x. □

Lemma 3.5. *Let G_m $(m<\omega)$ be groups in $\mathcal{S}$ and define $p_m^*(x) = \{p_m(H)\colon H \in x\}$. If X_{G_m} is non-empty, then $p_m^*(x)$ belongs to X_{G_m}.*

Proof. Let $i_m : G_m \to G$ be the canonical embedding, that is, i_m is the homomorphism satisfying $p_m i_m(g) = g$ and $p_k i_m(g) = e$ for $k \neq m$ and $g \in G$. To show that $p_m^*(x)$ contains an uncountable group by contradiction, we suppose the negation. Then $p_m(H)$ is finitely generated by Lemma 2.6. Since X_{G_m} is non-empty, we have an uncountable $K \in \mathcal{H}_{G_m}$. Since any finitely generated subgroups of G_m are compatible with K, $i_m(K)$ is compatible with any $F \Subset x$, which implies $i_m(K) \in x$ by the maximality of x. Hence $K = p_m i_m(K)$ belongs to $p_m^*(x)$, which is a contradiction.

Since $p_m^*(x)$ is compatible, now it suffices to show that $p_m^*(x)$ is maximal. Let $H' \in \mathcal{H}_{G_m}$ be compatible with any $F \Subset p_m^*(x)$. Since $i_m(H')$ is compatible with any $F \Subset x$ by Lemma 3.2, $i_m(H')$ belongs to x and consequently H' belongs to $p_m^*(x)$. □

Theorem 3.6. *Let G_m $(m < \omega)$ be groups in $\mathcal{S}$, $G = \prod_{m<\omega} G_m$ and $I = \{m < \omega\colon X_{G_m} \neq \emptyset\}$. Then the one of the following holds*:

(1) $X_G \simeq \prod_{m\in I} X_{G_m}$, *if I is non-empty;*
(2) X_G *is a one-point space, if I is empty and the set $\{m < \omega\colon G_m \neq \{e\}\}$ is infinite*;
(3) X_G *is empty, otherwise.*

Proof. We only prove the first case. We use the notion i_m in the proof of Lemma 3.5. Define $\varphi : X_G \to \prod_{m\in I} X_{G_m}$ by: $\varphi(x)(m) = p_m^*(x)$. Then φ is surjective and injective by Lemmas 3.2, 3.4 and 3.5. It now suffices to show that φ is continuous and inversely continuous. To see that each p_m^* is continuous, let $\lim_{n\to\infty} x_n = x$ in X_G and choose an uncountable $K_n \in p_m^*(x_n)$. Then $i_m(K_n) \in x_n$ by Lemma 3.2. There exists a sequence $(H_n\colon n < \omega)$ with $i_m(K_n) \preceq H_n \in x_n$ which witnesses $\lim_{n\to\infty} x_n = x$ and so we have $K_n \preceq p_m(H_n)$. Hence $p_m(H_n)$ is uncountable and belongs to $p_m^*(x_n)$. For $b_n \in p_m(H_n)$, choose $a_n \in H_n$ so that $p_m(a_n) = b_n$. Then there exists $h : \times_{n<\omega}\mathbb{Z}_n \to G$ such that $h(\delta_n) = a_n$ $(n < \omega)$. Now $p_m h(\delta_n) = b_n$ $(n < \omega)$ and $\mathrm{Im}(p_m h) \in p_m^*(x)$, which implies the continuity of p_m^*. Hence, φ is continuous. To see that φ is inversely continuous, suppose that $\lim_{n\to\infty} p_m^*(x_n) = p_m^*(x)$ for every $m \in I$. Let $H_n \in x_n$ be uncountable for each $n < \omega$. Since $p_m(H_n) \in p_m^*(x_n)$ we have a sequence $(K_{mn}\colon n < \omega)$ with $K_{mn} \in p_m^*(x_n)$ which witnesses $\lim_{n\to\infty} p_m^*(x_n) = p_m^*(x)$ for each $m \in I$. Then

$$\prod_{k\leqslant n, k\in I} K_{nk} \times \prod_{k\leqslant n, k\notin I} \{e\} \times \prod_{k>n} p_k(H_n) \in x_n \quad \text{and}$$

$$H_n \leqslant \prod_{k<\omega} p_k(H_k) \preceq \prod_{k\leqslant n, k\in I} K_{nk} \times \prod_{k\leqslant n, k\notin I} \{e\} \times \prod_{k>n} p_k(H_n)$$

hold by Lemmas 3.2 and 3.3. Let $a_n \in \prod_{k\leqslant n,k\in I} K_{nk} \times \prod_{k\leqslant n,k\notin I}\{e\} \times \prod_{k>n} p_k(H_n)$ for $n < \omega$. Fix m. Since $p_m(a_n) \in K_{mn}$ or $p_m(a_n) = e$ for $n \geqslant m$, there exists $h_m : \times_{n<\omega}\mathbb{Z}_n \to G_m$ such that $h_m(\delta_n) = p_m(a_n)$ $(n < \omega)$ and $\mathrm{Im}(h_m) \in p_m^*(x)$. Define $h : \times_{n<\omega}\mathbb{Z}_n \to G$ by: $p_m h = h_m$ $(m < \omega)$. Hence, we have $h(\delta_n) = a_n$ $(n < \omega)$ and $\mathrm{Im}(h) = \prod_{m<\omega} \mathrm{Im}(h_m) \in x$ by Lemma 3.2. □

4. Free products

We call a group G *quasi-atomic*, if for each homomorphism $h : G \to A * B$ there exists a finitely generated subgroup A' of A or B' of B such that $\mathrm{Im}(h)$ is contained in $A' * B$ or $A * B'$.

By definition, finitely generated groups and Abelian groups are quasi-atomic, but free products of infinitely generated groups are not quasi-atomic. Every homomorphic image of a quasi-atomic group is also quasi-atomic. This section is devoted to prove the next theorem.

Theorem 4.1. *Let G_i be a finitely generated groups for each $i \in I$. Then $\times^{\sigma}_{i\in I} G_i$ is quasi-atomic.*

For a word $W \in \mathcal{W}(G_n\colon n \in \omega)$ and $n \in \omega$, $l_n(W)$ is the number of appearances of elements of G_n in W. For $x \in *_{j\in J} H_j$, $l(x)$ denotes the length of the reduced word for x. Theorem 4.1 strengthens [3, Corollary 2.5] and the proof uses the content of [3, Section 2]. First we recall [3, Lemma 2.3]. We state it in a more precise form which follows from its proof.

Lemma 4.2 [3, Lemma 2.3]. *Let H_j $(j \in J)$ be groups. Let $m + n + 2 \leqslant k$ for $m, n, k \in \mathbb{N}$ and $u, x_i, z \in *_{j\in J} H_j$ $(1 \leqslant i \leqslant M)$. If $l(u) \leqslant m$, $u = x_1 z^k \cdots x_M z^k$ and $l(x_i) \leqslant n$ for all $1 \leqslant i \leqslant M$, then one of the following holds*:

(1) *z is a conjugate to an element of some H_j;*
(2) *$z = x^{-1} f x y^{-1} g y$ for some $f \in H_j$ and $g \in H_{j'}$ with $f^2 = g^2 = e$, and some $x, y \in *_{j\in J} H_j$ such that $x_i = z^p x^{-1} f x$ or $x_i = y^{-1} g y z^p$ for some i and p.*

Lemma 4.3 [3, Corollary 2.5]. *Let $h : \times_{n\in\omega} G_n \to *_{j\in J} H_j$ be a homomorphism for groups $G_n (n \in \omega)$ and $H_j (j \in J)$. If every G_i is finitely generated, then there exists $F \Subset J$ such that $h(\times_{n\in\omega} G_n) \leqslant *_{j\in F} H_j$.*

Definition 4.4. A subset C_1 of $A * B$ is the set of all conjugates to elements of $A \cup B$, i.e., $C_1 = \{x^{-1} u x\colon u \in A \cup B,\ x \in A * B\}$. Also let $C_2 = \{xy\colon x, y \in C_1\}$.

The proof of the next lemma is straightforward and is omitted.

Lemma 4.5. *The reduced word W for an element of $C_2 (\subseteq A * B)$ is one of the following forms*:

(1) *empty*;
(2) *$V^{-1} u V$ where $u \in A \cup B$ and V is a reduced word, that is, $W \in C_1$ as an element of $A * B$;*
(3) *$V_2^{-1} V_0^{-1} u_0 V_0 V_1^{-1} u_1 V_1 V_2$ where $u_0, u_1 \in A \cup B$ and V_0, V_1, V_2 are reduced words*;
(4) *$V_2^{-1} v_0 V_0^{-1} u_0 V_0 v_1 V_1^{-1} u_1 V_1 v_2 V_2$ where $u_0, u_1, v_0, v_1, v_2 \in A \cup B$ and V_0, V_1, V_2 are reduced words and $v_0 v_1 v_2 = e$.*

Lemma 4.6. *Let A' and B' be subgroups of A and B, respectively. If $a_0 \in A \setminus A'$, $b_0 \in B \setminus B'$ and $a_1 \in A \setminus \langle A' \cup \{a_0\}\rangle$, then*

(1) *$u^{-1} a_0 u v^{-1} b_0 v u^{-1} a_1 u \notin C_2$ for $u, v \in A' * B'$;*
(2) *$w u^{-1} a_0 u v^{-1} b_0 v u^{-1} a_1 u$ does not belong to*

$$\bigl\{\bigl(x^{-1} f x y^{-1} g y\bigr)^p x^{-1} f x,\ y^{-1} g y \bigl(x^{-1} f x y^{-1} g y\bigr)^p\colon\ f, g \in A \cup B,\ x, y \in A * B,\ p \geqslant 0\bigr\}$$

for $u, v, w \in A' * B'$.

Proof. Let U and V be reduced words for u and v, respectively. If the left most letter a' of U belongs to A', then ${a'}^{-1}a_0a' \in A \setminus A'$ and ${a'}^{-1}a_1a'$ belongs to $A \setminus \langle A' \cup \{a_0\}\rangle$. Hence, we may assume that $U^{-1}a_0U$, $V^{-1}b_0V$ and $U^{-1}a_1U$ are reduced. Let W be the reduced word of VU^{-1}. When W is empty or the left most letter of W^{-1} does not belong to A', the reduced word of $U^{-1}a_0W^{-1}b_0Wa_1U$ is of the form $U^{-1}a_0X^{-1}b_1Xa_1U$ for some $b_1 \in B \setminus B'$ and some word X. When the left most letter of W^{-1} belongs to A', the reduced word of $U^{-1}a_0W^{-1}b_0Wa_1U$ is of the form $U^{-1}a_2X^{-1}b_1Xa_3U$ for some $a_2 \in A \setminus A'$, $b_1 \in B \setminus B'$, $a_3 \in A \setminus \langle A' \cup \{a_2\}\rangle$ and some word X. In the both cases, they are not of the form indicated in Lemma 4.5 and the conclusion (1) holds by that lemma.

To show (2) by contradiction, suppose there exist $f, g \in A \cup B$, $x, y \in A * B$ and $p \geqslant 0$ such that $wu^{-1}a_0uv^{-1}b_0vu^{-1}a_1u = (x^{-1}fxy^{-1}gy)^p x^{-1}fx$ or $y^{-1}gy(x^{-1}fxy^{-1}gy)^p$. Consider the reduced form of $u^{-1}a_0uv^{-1}b_0vu^{-1}a_1u$ shown in the proof of (1). The arrangement of a_0, b_0, a_1 implies that $p \geqslant 1$. Since a_1 or its conjugate in A appears just one time in the reduced word for $wu^{-1}a_0uv^{-1}b_0vu^{-1}a_1u$, we get a contradiction. □

Proof of Theorem 4.1. First we prove the conclusion for a homomorphism $h : \times_{n<\omega}\mathbb{Z}_n \to A * B$. By Kurosh's Theorem [8, Section 34] or [7, Chapter 17], the image of h is a free product of copies of $\mathbb{Z}$ and conjugate groups of subgroups of A or B. By Lemma 4.3 the number of components of this free product is finite. To prove the conclusion by contradiction, suppose that there exists no A' nor B' required in the conclusion. Since the component isomorphic to a free group is finitely generated, there exist $u, v \in A * B$ and infinitely generated subgroups A^* and B^* such that $u^{-1}A^*u$ and $v^{-1}B^*v$ are free factors of $\mathrm{Im}(h)$. By our assumption neither A^* nor B^* is contained in any finitely generated subgroup A' of A nor B' of B. Note that one can find u, v, A^* and B^* with the above property for $h(\times_{n\geqslant m}\mathbb{Z}_n)$ as well. We choose $k_m < k_{m+1}$ and $x_m \in \times_{n\geqslant m}\mathbb{Z}_n$ by induction as follows:

We choose x_1 so that $h(x_1) \neq e$ and $k_1 = 2$. In the m-step, let $k_m = m + \max\{l(h(x_i \cdots x_{m-1}))\colon\ 0 \leqslant i \leqslant m-1\} + k_{m-1}$. Applying the above argument to $h(\times_{n\geqslant m}\mathbb{Z}_n)$, we obtain $u^{-1}A^*u$ and $v^{-1}B^*v$. Take finitely generated subgroups A' of A and B' of B such that $u \in A'$, $v \in B'$ and $h(x_i) \in A' * B'$ for every $0 \leqslant i \leqslant m-1$. By applying Lemma 4.6 to $A^* \setminus A'$ and $B^* \setminus B'$, we obtain $x_m \in \times_{n\geqslant m}\mathbb{Z}_n$ such that

(1) $h(x_m) \notin C_2$;
(2) $h(x_i) \cdots h(x_m)$ does not belong to

$$\big\{\big(x^{-1}fxy^{-1}gy\big)^p x^{-1}fx,\ y^{-1}gy\big(x^{-1}fxy^{-1}gy\big)^p\colon\ f, g \in A \cup B,\ x, y \in A * B,\ p \geqslant 0\big\}$$

for any $1 \leqslant i \leqslant m-1$.

Let W_m be the reduced word for x_m for each $1 \leqslant m < \omega$. We use the following notion in the proof of [3, Lemma 2.4]. Let Seq be the set of all finite sequences of natural numbers and denote the length of $s \in Seq$ by $lh(s)$. An element $s \in Seq$ is denoted by $\langle s_1, \ldots, s_n\rangle$ where $s_k \in \mathbb{N}$ $(1 \leqslant k \leqslant n)$. For $s, t \in Seq$, $s \prec t$ if $s(i) < t(i)$ for the minimal i with $s(i) \neq t(i)$ or t extends s.

Let $\overline{V} = \{(s,p)\colon\ s \in Seq,\ 1 \leqslant s(i) \leqslant k_i \text{ for } 1 \leqslant i < \omega,\ p \in \overline{W_i}\}$ with the lexicographical ordering and $V(s) = W_{lh(s)}(p)$. Then V is a word in $\mathcal{W}(\mathbb{Z}_n\colon\ n < \omega)$. Let $\overline{V_m} = \overline{V} \cap \{(s,p)\colon\ lh(s) \geqslant m,\ s_i = 1 \text{ for } 1 \leqslant i \leqslant m\}$ and $V_m = V \restriction \overline{V_m}$. Finally, choose N so that $N \geqslant l(h(V))$. Then $h(V)$ is equal to the element expressed in the following figure:

$$
\begin{array}{llll}
h(x_1) & h(x_2) & \cdots & h(x_{N-1})h(V_N)^{k_N} \\
 & & \ddots & \\
 & & & h(x_{N-1})h(V_N)^{k_N} \\
 & h(x_2) & \cdots & h(x_{N-1})h(V_N)^{k_N} \\
 & & \ddots & \\
 & & & h(x_{N-1})h(V_N)^{k_N} \\
 & \vdots & & \\
 & & \ddots & \\
 & & & h(x_{N-1})h(V_N)^{k_N} \\
h(x_1) & h(x_2) & \cdots & h(x_{N-1})h(V_N)^{k_N} \\
 & & \ddots & \\
 & & & h(x_{N-1})h(V_N)^{k_N} \\
 & h(x_2) & \cdots & h(x_{N-1})h(V_N)^{k_N} \\
 & & \ddots & \\
 & & & h(x_{N-1})h(V_N)^{k_N} \\
 & \vdots & & \\
 & & \ddots & \\
 & & & h(x_{N-1})h(V_N)^{k_N}.
\end{array}
$$

Applying Lemmas 4.2 and 4.6 we conclude $h(V_N) \in C_1$ and by the same argument $h(V_{N+1}) \in C_1$, which implies $h(V_{N+1})^{-k_{N+1}} \in C_1$. Since $x_N = V_N V_{N+1}^{-k_{N+1}}$, we have $h(x_N) \in C_2$, which contradicts the construction.

Next we treat with the case $h : \times^{\sigma}_{i \in I} \mathbb{Z}_i \to A * B$. We modify the above construction as follows: We choose countable $J_m \subseteq I$ and finite $S_m \subseteq I$ in addition to x_m and $k_m < k_{m+1}$ by induction so that $J_m \cap S_{m-1} = \emptyset$ and $x_m \in \times_{i \in J_m} \mathbb{Z}_i$ and $\bigcup_{m<\omega} J_m = \bigcup_{m<\omega} S_m$. Then we apply the above procedure taking infinitely generated subgroups A^* and B^* for $\times_{i \in I \setminus S_{m-1}} \mathbb{Z}_i$ and we can define V as before and get a contradiction.

Since $\times^{\sigma}_{i \in I} F_i$ is isomorphic to $\times^{\sigma}_{j \in J} \mathbb{Z}_j$ for finitely generated free groups F_i, $\times^{\sigma}_{i \in I} G_i$ is a homomorphic image of $\times^{\sigma}_{j \in J} \mathbb{Z}_j$ and the conclusion holds. $\square$

Corollary 4.7. *Let A and B are groups in $\mathcal{S}$. Then X_{A*B} is the topological sum of X_A and X_B.*

Proof. Let G be the free product $A * B$ and $H \in \mathcal{H}_G$ be an uncountable subgroup. By Theorem 4.1 we first assume that there exists $F_0 \Subset A$ such that $H \leqslant \langle F_0 \rangle * B$. As in the proof of Theorem 4.1, H is isomorphic to a finite free product of copies of $\mathbb{Z}$ and conjugate groups of subgroups of A or B. Every letters in A and subgroups of A which appears in this presentation is in $\langle F_0\rangle$ and there exists an uncountable subgroup H_0 of B such that H is a free product of a conjugate subgroup of H_0 and a finitely generated group. Here $H_0 \in \mathcal{H}_B$. We have $F_1 \Subset B$ such that $H \leqslant \langle H_0 \cup F_0 \cup F_1\rangle$. We have a similar conclusion when A is replaced by B.

By the preceding fact we easily see that the only one of $\{H \in x\colon H \in \mathcal{H}_A\} \in X_A$ and $\{H \in x\colon H \in \mathcal{H}_B\} \in X_B$ holds for $x \in X_G$. On the other hand we also easily see that $\{\langle H \cup F\rangle\colon F \Subset A \cup B,\ H \in x\} \in X_G$ for each $x \in X_A \cup X_B$. Let $Y_A = \{x \in X_G\colon \{H \in x\colon H \in \mathcal{H}_A\} \in X_A\}$ and $Y_B = \{x \in X_G\colon \{H \in x\colon H \in \mathcal{H}_B\} \in X_B\}$. Then X_G is the disjoint union of Y_A and Y_B and Y_A and Y_B are homeomorphic to X_A and X_B, respectively.

To show that Y_A is closed in X_G, suppose that $\lim_{n\to\infty} x_n = x$ for x_n's in Y_A. Take uncountable $H_n \in x_n$. We have a sequence $(K_n\colon n < \omega)$ with $H_n \preceq K_n \in x_n$ which witnesses $\lim_{n\to\infty} x_n = x$. Since K_n is uncountable, K_n contains a conjugate subgroup to an uncountable subgroup of A. We choose $a_n \in K_n$ so that $a_n \neq e$ and a conjugate to an element of A. We remark $p_A(a_n) \neq e$ and $p_B(a_n) = e$. Then we have $h : \circledast_{n<\omega}\mathbb{Z}_n \to G$ such that $h(\delta_n) = a_n$ $(n < \omega)$. Since $p_B(a_n) = e$ for every $n < \omega$, we have $p_B h(x) = e$ for every $x \in \circledast_{n<\omega}\mathbb{Z}_n$ by Lemma 2.5. By the proof of Lemma 2.6 $\mathrm{Im}(h)$ is uncountable, which implies $\mathrm{Im}(h)$ contains an subgroup conjugate to an uncountable subgroup of A. Therefore x belongs to Y_A. We obtain the corresponding statements also for Y_B. □

5. Recovering a space

A point $x \in X$ is called a *wild* point, if X is not locally semi-simply connected at x. X^w denotes the subspace consisting of all wild points of X. A space X is called *wild*, if $X = X^w$ holds.

Theorem 5.1. *Let X be a locally path-connected, path-connected, one-dimensional metric space and G be the fundamental group $\pi_1(X)$. Then X_G is homeomorphic to X^w. Consequently, in addition if X is wild, X_G is homeomorphic to X itself.*

Proof. As we stated in the proof of Proposition 2.1, each uncountable $H \in \mathcal{H}_G$ determines a unique point $x^* \in X$ and also determines a unique point $x \in X_G$. Moreover H is conjugate to the image of a homomorphism induced by a continuous map $f : (\mathbb{H}, o) \to (X, x^*)$, where o is the unique wild point of $\mathbb{H}$. Hence x^* is a wild point. This correspondence $x \mapsto x^*$ is a bijection, since H and H' are compatible only if the points of X uniquely determined by H and H' are the same for uncountable $H, H' \in \mathcal{H}_G$. To see the continuity, let $Y^* = \{y^*\colon y \in Y\}$ for $Y \subseteq X_G$. It suffices to show that Y^* is closed if and only if Y is closed. This easily follows from the fact that for $x_n \in X_G$ $(*)$ holds if and only if $\lim_{n\to\infty} x_n^* = x^*$ by [2, Theorem 1.1]. □

Together with Theorem 3.6 we have,

Corollary 5.2. *Let X_n be locally path-connected, path-connected, one-dimensional metric spaces such that $X_n^w \neq \emptyset$ and G be the fundamental group $\pi_1(\prod_{n<\omega} X_n) \simeq \prod_{n<\omega} \pi_1(X_n)$.*

Then X_G is homeomorphic to $\prod_{n<\omega} X_n^w$. Consequently, in addition if $X_n = X_n^w$ for every $n < \omega$, X_G is homeomorphic to $\prod_{n<\omega} X_n$.

Remark 5.3. Let each X_n be a copy of one-dimensional wild Peano continuum and $Y = \prod_{n<\nu} X_n$, where $\nu \leqslant \omega$. Then $X_{\pi_1(Y)} \simeq Y$ by Corollary 5.2, but the topology of $X_{\pi_1(Y)}$ does not coincide with the one defined by the condition (T2). This can be seen as follows. Take $H_n \in y_n$ to be subgroups of $\pi_1(X_0) \times \{e\}$, where e is the identity of $\pi_1(\prod_{1\leqslant n<\nu} X_n)$. Then the satisfaction of (T2) is the information only on the first co-ordinate and so (T2) does not give the topology of Y.

Concerning (T1) we have no examples which show the difference between the topologies $(*)$ and (T1). Since we feel some difficulty to get Theorem 3.6 and Corollary 4.7 based on the definition (T1), we adopt $(*)$.

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